that the vortex core region centers move radially outward along a line at an angle of 67° from a horizontal axis passing through the center of the test configuration. The data taken here were not found to exhibit such a trent. The core region center moves radially outward along a line curving away from the center of the test configuration. The outward curve becomes sharper as the angle of attack is increased. This trend is exactly opposite to the supersonic case as reported in Ref. 7. That is, in the supersonic case the core region centers were found to first move radially outward then to move along a line curving slightly back toward the center of the test configuration.

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A New Integral Calculation of Skin Friction on a Porous Plate

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Nomenclature

C_f = skin-friction coefficient, $\tau_w/\frac{1}{2}\rho u_0^2$
f = velocity profile, u/u_0
k = coefficients of the polynomial velocity profiles
Re_x = Reynolds number based on distance x, u_0x/ν
Re_δ = Reynolds number based on boundary-layer thickness, $u_0\delta/\nu$
(u,v) = velocity components corresponding to (x,y)
(x,y) = Cartesian coordinate system with origin at the leading edge, x along the freestream direction
α = $\int_0^1 f d\eta - \int_0^1 f^2 d\eta - \int_0^1 f d\eta \int_0^\eta f d\eta + \int_0^1 d\eta \int_0^\eta f^2 d\eta$
β = $1 - \int_0^1 f d\eta$
γ = $\int_0^1 f d\eta - \int_0^1 f^2 d\eta$
δ = boundary-layer thickness
ϵ = blowing parameter, v_w/u_0
η = similarity variable, y/δ
λ = blowing parameter, $(v_w/u_0)(x/\delta)$
$\bar{\lambda}$ = blowing parameter, $(v_w/u_0)(Re_x^{1/2})$
ν = kinematic viscosity
ρ = density of the fluid
τ = shear stress

Subscripts

0 = freestream condition
w = wall

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Introduction

THIS Note reports on some preliminary results for the incompressible, laminar skin friction on a porous plate calculated by a refined Karman-Pohlhausen (K-P) method. Only two types of surface blowing (or suction) are considered, namely, the similarity type for which $v_w \sim x^{-1/2}$ and the uniform type for which v_w = constant. Exact numerical results exist for both cases so that the capability of this new method can be tested. It will be revealed that remarkably accurate and reliable results can be obtained in analytic forms, requiring only simple and elementary calculations.

Volkov[1] recently advanced the basic idea of the refined K-P method, and a slight modification is made here to generalize application to flows over permeable surfaces in order to assess its validity in general boundary-layer flows including the effects of surface mass transfer. In essence, the method is based on a double integration of boundary-layer equations in the direction normal to the external main stream. The skin-friction term can thus be expressed in terms of certain integrals of the assumed velocity profile instead of the derivative of it, as in the usual K-P method. One expects that the accuracy of the skin-friction, so expressed, should be significantly increased and that, as a consequence, the method should yield improved results in cases where the skin friction plays an important role, such as boundary-layer flows pertinent to separation.

Analysis

We begin with the integrated version of the laminar boundary-layer equations over a porous plate, i.e.,

$$\frac{\partial}{\partial x}\int_0^y u^2 dy + uv = \nu\frac{\partial u}{\partial y} - \frac{\tau_w(x)}{\rho} \tag{1}$$

Note that Eq. (1) with the upper limit of the integral replaced by $\delta(x)$ is the familiar K-P equation for the determination of $\delta(x)$. However, following the idea of Volkov's refinement,[1] we shall use this equation as an expression for the skin friction, $\tau_w(x)$. The quantity $\delta(x)$ will then be determined by an equation obtained by another integration of Eq. (1). Thus the ordinary differential equation determining the basic quantity $\delta(x)$ is

$$\int_0^\delta dy \frac{\partial}{\partial x}\int_0^y u^2 dy + v_w\int_0^\delta u dy - \int_0^\delta u dy \frac{\partial}{\partial x}\int_0^y u dy = \nu u_0 + \delta\frac{d}{dx}\int_0^\delta u^2 dy + \delta u_0 v_w - \delta u_0\frac{d}{dx}\int_0^\delta u dy \tag{2}$$

Note that in arriving at Eq. (2), we have substituted for $\tau_w(x)$ an expression derived from Eq. (1), i.e.,

$$\frac{1}{2}C_f \equiv \frac{\tau_w(x)}{\rho u_0^2} = \frac{d}{dx}\int_0^\delta \frac{u}{u_0}\left(1 - \frac{u}{u_0}\right)dy - \epsilon \tag{3}$$

Eqs. (2) and (3) form the basis of the ensuing calculations.

As for the velocity profiles to be used in Eq. (2), we shall adopt the familiar form of polynomials with constant coefficients, i.e., we assume

$$u/u_0 = f(\eta) = \Sigma_i k_i\eta^i, \quad k_i = \text{const} \tag{4}$$

This choice of the profiles presumes the existence of similarity of solution. While the similarity property is indeed preserved in the case of similarity blowing (or suction), it is certainly destroyed in the case of uniform blowing (or suction). Further, these profiles preclude the possibility of satisfying the basic partial differential equation at the wall where a nonvanishing normal velocity exists. Nevertheless, this family of profiles is still used in the following calculation. The purpose is to test the capability of the refined method in coping with the improperly chosen profiles as well as to simplify the calculations. It should be pointed out that the use of polynomials with variable coefficients which would enable the differential equation to be satisfied at the wall causes no essential difficulties; it only complicates the calculation somewhat.

With the profiles as chosen, Eqs. (2) and (3) reduce, respectively, to the following form:

$$\alpha\delta\, d\delta/dx = \frac{\nu}{u_0} + \beta\delta v_w/u_0 \tag{5}$$

and

$$(\tfrac{1}{2})C_f = (\gamma/\alpha)(Re_\delta)^{-1} + [\beta(\gamma/\alpha) - 1]\epsilon \tag{6}$$

where α, β and γ are profile parameters defined in the Nomenclature.

Two profiles, namely, $f = \eta$ and $f = 2\eta - 2\eta^3 + \eta^4$ have been used in the present calculations, and the corresponding profile parameters are $(\alpha,\beta,\gamma) = (\frac{1}{8}, \frac{1}{2}, \frac{1}{6})$, (0.06136, 0.3, 0.11747), respectively.

Applications and Results

Similarity blowing (or suction)

It is anticipated that for this type of flow, $\delta \sim x^{1/2}$. Therefore, we write

$$v_w/u_0 = \lambda(\delta/x) \tag{7}$$

where the constant λ is obviously a blowing (or, if negative, suction) parameter.

Then Eqs. (5) and (6) yield readily the following closed-form results:

$$\delta/x = [Re_x(\alpha/2 - \beta\lambda)]^{-1/2} \tag{8}$$

and

$$(\tfrac{1}{2})C_f(Re_x)^{1/2} = (\alpha/2 - \beta\lambda)^{-1/2}(\gamma/2 - \lambda) \tag{9}$$

The relation between the conventional blowing parameter, $\tilde{\lambda} \equiv \epsilon Re_x^{1/2}$, and λ can easily be found to be

$$\tilde{\lambda} = \lambda(\alpha/2 - \beta\lambda)^{-1/2} \tag{10}$$

The results of $(\frac{1}{2})C_f Re_x^{1/2}$ as a function of $\tilde{\lambda}$ are plotted in Figs. 1 and 2 (magnified scale) for suction and blowing, respectively. Exact solutions are taken from Ref. 2.

Applying the limit process of $\lambda \to -\infty$ with Re_x fixed to Eqs. (8) and (9), one can easily show that all the profile parameters drop out of the skin-friction expression, and that the expected limiting solution of asymptotic suction, i.e., $(\frac{1}{2})C_f = |\epsilon|$, is deduced.

On the other hand, the critical value of $\tilde{\lambda}$ corresponding to blow-off is found to be profile-dependent. It has the value of 0.578 and 0.513 for the two profiles used, as compared to the exact value of 0.619.[2]

Uniform blowing (or suction)

Exact solutions for suction were given by Iglisch[3] for suction and by Lew and Fanucci[4] and Catherall et al.[5] for blowing. In our calculation, Eqs. (5) and (6) are easily solved in closed-form for ϵ = constant.

The result can be cast into the following convenient form:

$$(\epsilon Re_x^{1/2})^2/\epsilon Re_\delta = \frac{\alpha}{\beta}\left[1 - \frac{1}{\epsilon Re_\delta}\frac{1}{\beta}\log(1 + \beta\epsilon Re_\delta)\right] \tag{11}$$

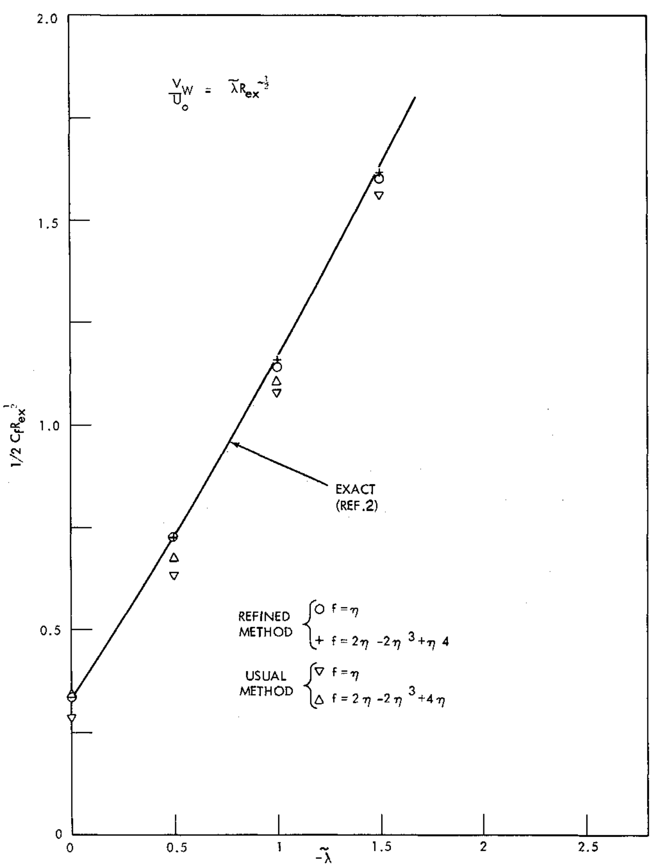

Fig. 1 Skin friction for similarity suction.

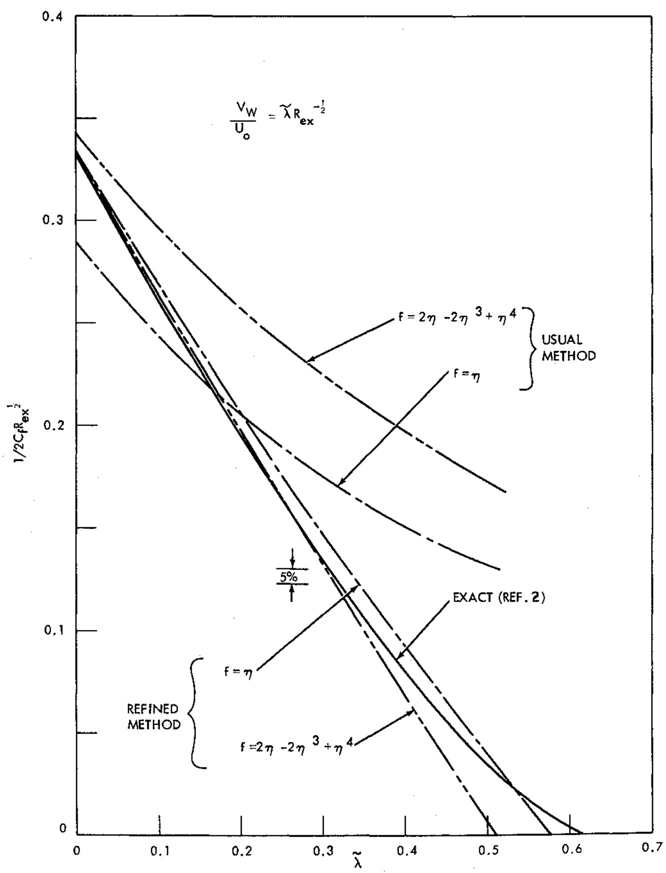

Fig. 2 Skin friction for similarity blowing.

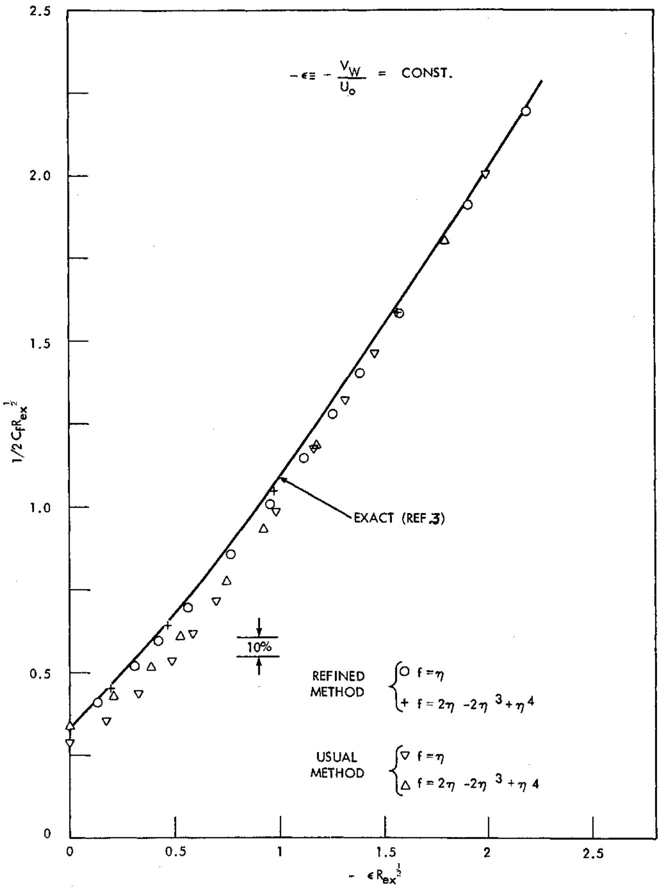

Fig. 3 Skin friction for uniform suction.

and

$$(\tfrac{1}{2})C_f Re_x^{1/2} = \frac{\gamma}{\alpha}(\epsilon Re_x^{1/2})/(\epsilon Re_\delta) + \left(\beta\frac{\gamma}{\alpha} - 1\right)\epsilon Re_x^{1/2} \tag{12}$$

Results are plotted in Figs. 3 and 4 for suction and blowing (magnified scale), respectively, along with the exact solutions.

Application of the limit process of $\epsilon \to -\infty$, $Re_\delta \to 0$ with $\beta\epsilon Re_\delta \to -1$ again leads to the asymptotic suction result with all the profile parameters dropped out. The blow-off point is profile-dependent, and is found to be at $\epsilon Re_x^{1/2} = 0.58$ and 0.67, respectively, for the two profiles used, as compared to the numerical result of 0.86.[5]

To highlight the degree of improvement due to the present refinement, the same simple profiles have been employed in the usual K-P method to calculate the skin friction for the two cases considered. Only the results are presented in the figures. One sees that although the asymptotic suction limit is correctly predicted, the results in the mild suction and the entire blowing regime are clearly erroneous, particularly the prediction of blow-off.

Discussion and Concluding Remarks

The results obtained from the present new method are clearly in very good agreement with exact numerical solutions from suction up to fairly strong blowing, indicating that Volkov's idea indeed works for boundary layers with mass transfer as well.

For the similarity v_w case, the accuracy of the results for $C_f Re_x^{1/2}$ is found to be within 2% in the entire suction region, and about 7% for $\bar{\lambda}$ as high as 0.3, where the value of $C_f Re_x^{1/2}$ itself becomes very small. For the case of constant v_w, the results appear to be equally encouraging. The accuracy achieved is within 3% throughout the suction region up to $\epsilon Re_x^{1/2} \approx 0.3$. For both cases, the results reduce to the asymptotic suction limit analytically for large suctions. Considering the crude nature of the profiles and the remarkable simplicity in the calculations, these results are indeed surprisingly good.

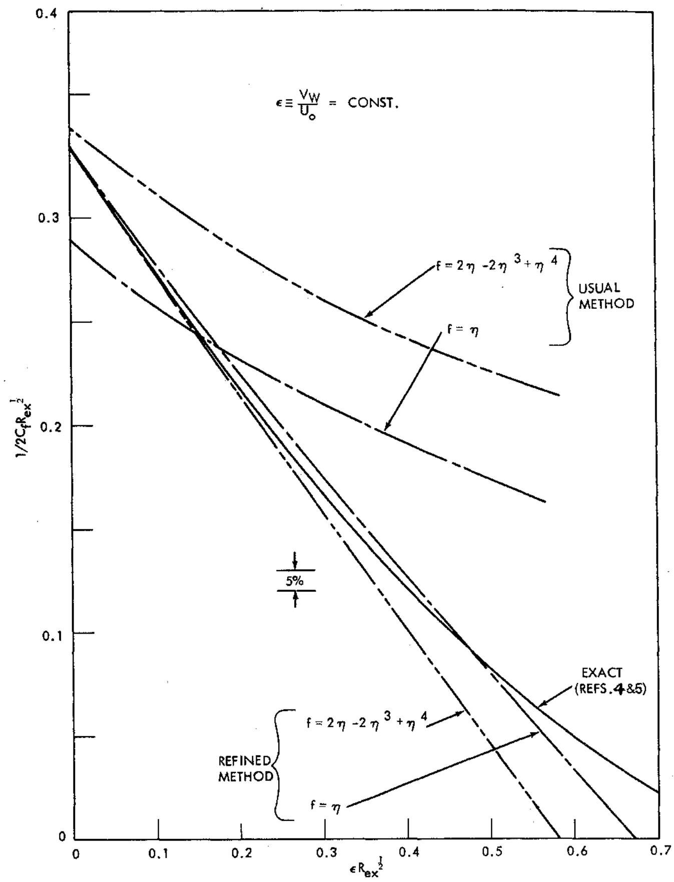

Fig. 4 Skin friction for uniform blowing.

Of equal importance is the insensitivity of the results from the refined method to the choice of the profiles, as shown in the results. This is evidently connected with the use of an integral expression for the skin friction. This insensitivity may be interpreted as the reliability of the method and is obviously of great value in all the integral methods.

In conclusion, we state that Volkov's idea of refining the K-P method proves to be very useful in boundary layers with mass transfer also. The merit of the refined method as reflected by the results presented lies in the simplicity in calculations, accuracy of the results, and the insensitivity of the results to the choice of the velocity profiles.†

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† Results of recent refined calculations[6] using variable-coefficient polynomial profiles further substantiate these conclusions.